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Conformal harmonic analysis on hyperboloids

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Let H be a hyperboloid of even dimension. We give a $\bar{\partial}$-cohomological interpretation of the decomposition of $L^2(H)$ on the parts corresponding to the discrete and continuous spectra.

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Let $H=H_{p,q-1}$ be the hyperboloid of signature $(p, q-1)$ in $\mathbb{R}^n$, $n=p+q-1$,

$$\lambda(x)=x_1^2+x_2^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_n^2=1\ . \tag{1}$$

Consider the canonical action of $SO(p, q-1)$ on H. Then

$$d\mu=d\lambda(x)\rfloor dx_1\wedge\cdots\wedge dx_n \tag{2}$$

is the invariant measure,

$$d\mu=\frac{dx_2\wedge\cdots\wedge dx_n}{2|x_1|}\quad \text{if } x_1\neq 0\ .$$

Consider the Hilbert space $L^2(H, d\mu)$ and the unitary representation U of $SO(p, q-1)$ in $L^2(H, d\mu)$. The spectral analysis of this representation is well known (see, e.g., ref. [1]). In this paper we want to discuss two new observations for odd n.

(I) If U_d and U_c are the parts of U corresponding to the discrete and continuous spectra, then U_d, U_c admit an extension to the irreducible unitary representations $\tilde{U}_d$, $\tilde{U}_c$ of $SO(p, q)$.

(II) $\tilde{U}_d$, $\tilde{U}_c$ can be realized in Hardy spaces of $\bar{\partial}$-cohomology for pseudo Hermitian symmetric spaces with boundary H. These spaces of cohomology admit a description in the language of Fourier integrals.

Both constructions can be generalized to many other affine symmetric spaces.

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1. Projectivization of the problem

Let PH be the projectivization of H, so that PH is defined in $\mathbb{RP}^n$ with homogeneous coordinates $x=(x_0, x_1, ..., x_n)$ by the equation

$$Q(x)=x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_n^2-x_0^2=0\,; \tag{3}$$

H is the affinization of PH corresponding to the hyperplane at infinity $x_0=1$.

Proposition 1. *The representation U of* $\mathrm{SO}(p, q-1)$ *can be extended as a unitary representation U of* $\mathrm{SO}(p, q)$.

Proof. Let

$$\mathrm{d}\tilde{\mu}=\mathrm{d}Q(x)\,\rfloor\,\omega(x)\,,$$

$$\omega(x)=\sum_{j=0}^{n}(-1)^j x_j \bigwedge_{i\neq j}\mathrm{d}x_i\,, \tag{4}$$

and $\mathscr{L}=\mathrm{L}^2(PH, \mathrm{d}\tilde{\mu})$ be the Hilbert space of homogeneous functions $f(x)$ of degree $-(n-1)/2$,

$$f(\lambda x)=|\lambda|^{-(n-1)/2}f(x)\,,\quad \lambda\in\mathbb{R}\,,$$

with the norm

$$\|f\|^2=\int_{PH}|f(x)|^2\,\mathrm{d}\tilde{\mu}\,. \tag{5}$$

In formula (5) we integrate over an arbitrary section of $\mathbb{R}^{n+1}\to\mathbb{RP}^n\supset PH$ and the integral is independent of the choice of section. The elements of $\mathrm{L}^2(PH, \mathrm{d}\tilde{\mu})$ are defined uniquely by their restrictions to the hyperplane $x_0=1$; these restrictions are elements of $\mathrm{L}^2(H, \mathrm{d}\mu)$. So

$$\mathscr{L}=\mathrm{L}^2(PH, \mathrm{d}\tilde{\mu})\cong\mathrm{L}^2(H, \mathrm{d}\mu)$$

and the natural representation of $\mathrm{SO}(p, q)$ in $\mathrm{L}^2(PH, \mathrm{d}\tilde{\mu})$ by translations gives an extension of the representation U; we will denote this extension also by U. □

2. Parabolic affinization and decomposition of U

Between affine forms of PH there are two hyperboloids $H_{p,q-1}$, $H_{p-1,q}$ and one paraboloid $\mathscr{P}_{p-1,q-1}$. Represent $Q(x)$ in the form

$$Q(x)=x_0x_1+x_2^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_n^2$$
$$=x_0x_1+m(x')\,,\quad x'=(x_2, ..., x_n)\,.$$

Then we get the paraboloid $\mathscr{P}=\mathscr{P}_{p-1,q-1}$ in the hyperplane $x_0=1$. The group $G_{\mathscr{P}}$ of affine automorphisms of $\mathscr{P}$ is $\mathbb{R}^{n-1} \rtimes (\mathbb{R}^{\times} \times \mathrm{SO}(p-1,q-1))$,

$$x' \mapsto gx' , \quad g \in \mathrm{SO}(p-1,q-1) , \quad x_1 \mapsto x_1 ;$$

$$x' \mapsto \lambda x' , \quad x_1 \mapsto \lambda^2 x_1 , \quad \lambda \in \mathbb{R} \setminus 0 ;$$

$$x' \mapsto x'+a , \quad x_1 \to x_1 - 2m(x', a) - m(a) , \quad a \in \mathbb{R}^{n-1} ,$$

$m(a, b)$ is the bilinear form corresponding to $m(x')$.

Lemma. *The restriction $U|_{G_{\mathscr{P}}}$ is the sum of two irreducible representations.*

Proof. Let us take $x_0=1$, and take $x'=(x_2, ..., x_n)$ as coordinates on $\mathscr{P}$. This gives the embedding $\mathscr{P} \cong \mathbb{R}^{n-1}_{x'} \to PH$. The image of $\mathrm{L}^2(PH, \mathrm{d}\tilde{\mu})$ under the restriction to $\mathbb{R}^{n-1}_{x'}$ coincides with $\mathrm{L}^2(\mathbb{R}^{n-1}_{x'}, \mathrm{d}x')$. Let us consider the Fourier transform,

$$f(x') \mapsto \mathscr{F}f = \tilde{f}(\xi') = \int_{\mathbb{R}^{n-1}} f(x') \exp(x_2\xi_2 + \cdots + x_n\xi_n) \, \mathrm{d}x' ,$$

and let $\tilde{\mathscr{L}}$ denote the image of $\mathscr{L}$. The action of $\mathbb{R}^{\times} \times \mathrm{SO}(p-1,q-1)$ on $\tilde{\mathscr{L}}$ is the same as on $\mathscr{L}$ and the elements $a \in \mathbb{R}^{n-1}$ will act as multiplication by $\exp(\mathrm{i}\langle a, \xi' \rangle)$. Let

$$\tilde{\mathscr{L}} = \tilde{\mathscr{L}}_+ \oplus \tilde{\mathscr{L}}_- ,$$

where $\tilde{\mathscr{L}}_\pm$ are the subspaces of elements of $\tilde{\mathscr{L}}$ with supports in the cones $V_\pm = \{\xi' : m(\xi') \gtrless 0\}$, respectively. It is clear that they, as well as their pre-images $\mathscr{L}_\pm$, are invariant and irreducible. So

$$\mathscr{L} = \mathscr{L}_+ \oplus \mathscr{L}_- ,$$

where $f \in \mathscr{L}_\pm$ iff $\mathrm{supp}\, \tilde{f} \subset V_\pm$ and $U|_{\mathscr{L}_\pm}$ are irreducible. □

3. Projectors on $\mathscr{L}_\pm$ and SO(p, q) invariance

The projectors $P_\pm$ on $\mathscr{L}_\pm$ are

$$P_\pm f = K_\pm * f ,$$

where $K_\pm = \mathscr{F}^{-1}\kappa(V_\pm)$, $\kappa(V_\pm)$ is the characteristic function of $V_\pm$. We will give explicit formulas for $K_\pm$.

Proposition. *We have*

$$K_+ + K_- = \delta(x')$$

and

(i) if p is odd, q is even then

$$K_+(x') = (-1)^{q/2}\pi^{-(n+1)/2}\Gamma((n-1)/2)(m(x'))_+^{-(n-1)/2};$$

(ii) if p and q are odd, then

$$K_+(x') = \tfrac{1}{2}\delta(x') + (-1)^{p-1}m^{(n-1)/2-1}(\partial/\partial x')\delta(m(x'));$$

(iii) if p is even, q is odd, then

$$K_+(x') = \delta(x') + (-1)^{p/2}\pi^{-(n+1)/2}\Gamma((n-1)/2)(m(x'))_-^{-(n-1)/2};$$

(iv) if p and q are even, then

$$K_+(x') = \tfrac{1}{2}\delta(x') + \tfrac{1}{2}(-1)^{p/2}\Gamma((n-1)/2)\pi^{(n+1)/2} \times (m_+^{[-(n-1)/2]} + (-1)^{(n-1)/2}m^{[-(n-1)/2]}).$$

We use the notation $m_\pm^\lambda$ for the meromorphic continuation (on λ) of the distribution

$$(m_\pm^\lambda, \varphi) = \int_{m \gtrless 0} |m|^\lambda \varphi \, \mathrm{d}x',$$

and $m_\pm^{[\lambda^0]}$ for the constant term in the Laurent series for $m_\pm^\lambda$ at $\lambda = \lambda^0$ [2].

This formula immediately follows from the formulas in ref. [2], ch. IV, 32, for distributions related to quadratic forms. For us it is important that $K_\pm$ are invariant distributions for $\mathbb{R}^\times \times \mathrm{SO}(p-1, q-1)$ of degree $-(n-1)$ so that they have the structure $\rho(m(x))$ where ρ is a homogeneous distribution in one variable.

Let us remark that, if $x_0 = y_0 = 1$ and $Q(x) = Q(y) = 0$, then

$$-Q(x, y) = m(x' - y'),$$

where

$$Q(x, y) = \tfrac{1}{2}(Q(x+y) - Q(x) - Q(y))$$

is the bilinear form corresponding to $Q(x)$. Therefore, if n is odd then $\mathscr{L} = \mathrm{L}^2(PH, \mathrm{d}\tilde{\mu})$,

$$P_\pm f(g) = \int_{\mathrm{P}^n} \rho(Q(x; y))\delta(Q(y))f(y)\omega(y) \qquad (6)$$

(equality of distributions).

The kernel $\rho(Q(x, y))$ is $\mathrm{SO}(p, q)$ invariant and $P_\pm$ commute with U:

Theorem 2. *The subspaces* $\mathscr{L}_\pm$ *are invariant under* $\mathrm{SO}(p, q)$ *if* n *is odd and the corresponding representations* $U_\pm$ *are irreducible.*

Remarks 3.

(i) We have defined SO(p, q)-invariant (projective) objects, $U_{\pm}$, using the Fourier integral, which is only $G_{\mathscr{P}}$ invariant (affine). If we replace in the definition the Fourier transform by the Radon transform we obtain projective invariant definitions.

Namely, let

$$\hat{f}(\xi, \eta) = \int_{\mathbb{R}P^n} f(x)\delta(Q(x))\delta(\langle \xi, x\rangle)\langle \eta, x\rangle^{-(n-3)/2}\omega(x) . \tag{7}$$

The integrand is homogeneous of degree 0 in x. Let

(i) $\eta = (\partial Q/\partial x)(x^0)$, $Q(x^0)=0$,

(ii) $\langle \xi, x^0\rangle = 0$.

If $x^0 = (0, 1, 0, \ldots, 0)$, $\langle \eta, x\rangle = x_0$, we have the usual Radon transform for $\varphi(x') = f(1, -m(x'), x')$ and the condition $f \in \mathscr{L}_{\pm}$ is equivalent to

$$\hat{f}(\xi, \eta) = 0 \quad \text{if } Q(\xi) \gtrless 0 . \tag{R$_\pm$}$$

The group SO(p, q) acts by translations of (ξ, η). One can give a direct description of functions $f(\xi, \eta)$ (an analog of the Paley–Wiener theorem for transformation (7)) and this gives an alternative realization of the representations $U_{\pm}$. We will discuss these results in a separate paper about integral geometry on hyperboloids.

(ii) On the hyperboloid $H_{p,q-1}$ there is a canonical conformal structure: isotropic cones are the intersections of $H_{p,q-1}$ with tangent hyperplanes. Then SO$(p, q-1)$ is the full affine conformal group and SO(p, q) is the full conformal group. The isotropic cones on the paraboloid $\mathscr{P}_{p-1,q-1}$ in coordinates x' are translation equivalent so that the conformal structure is flat. On the analytical level it gives us the possibility of using the Fourier integral for conformal constructions.

4. Pseudo-Hermitian symmetric manifolds on $\mathbb{C}PH$ and Hardy spaces of $\bar{\partial}$-cohomology

Let $\mathbb{C}PH$ be the complexification of PH in $\mathbb{C}P^n$,

$$Q(z) = z_0 z_1 + m(z') = 0 , \quad z = (z_0, z_1, z') \in \mathbb{C}P^n ,$$

$$Q(z, w) = z_0 w_1 + z_2 w_2 + \cdots + z_q w_q - z_{q+1} w_{q+1} - \cdots - z_n w_n .$$

Let us consider in $\mathbb{C}PH$ the domains

$$\mathscr{D}_{\pm} = \{ z \in \mathbb{C}PH : \operatorname{Re} Q(z, \bar{w}) \gtrless 0 \} . \tag{8}$$

Then for $z_0 = 1$, (8) is equivalent (direct computation) to

$$m(\operatorname{Im} z') \gtrless 0 . \tag{8$'$}$$

That is, in these coordinates, $\mathscr{D}_{\pm}$ are tube domains,

$$T_{\pm} = \mathbb{R}^{n-1} + \mathrm{i} V_{\pm} .$$

Domains $D_+ \supset T_+$ do not coincide if $p>2$: $\mathscr{D}_+$ contains a set on the plane $z_0=1$. We will show that $U_{\pm}$ admit realizations in Hilbert(–Hardy) spaces of cohomology $H_2^{(p-2)}(T_+), H_2^{(q-2)}(T_-)$.

Remark. If $p=2$ then T_+ has two components T_{++}, T_{+-}, corresponding to the two convex components of the cone V_+ and $H_2^{(0)}$ is the direct sum of the Hardy spaces of holomorphic functions on T_{++}, T_{+-}.

To define the cohomology we will construct coverings of tube domains $T_{\pm}$ by tube domains with convex bases. Let $x'=(r, u)$, $r=(x_2, ..., x_p)$, $u=(x_{p+1}, ..., x_n)$ and

$$V^{\omega} = \{(r, u) \in \mathbb{R}^{n-1} : \langle \omega, r\rangle^2 - u^2 > 0 , \quad \langle \omega, r\rangle > 0\} ,$$
$$\omega \in \mathrm{S}^{p-2} (\|\omega\| = 1) . \tag{9}$$

This is the convex cone which contains the $(p-2)$-space $\{(r, u) : \langle \omega, r\rangle = 0, u=0\}$. It is clear that

$$V^{\omega} \subset V_+ , \quad V_+ = \bigcup_{\omega \in \mathrm{S}^{p-2}} V^{\omega} , \tag{10}$$

so that the tube domains $T^{\omega} = \mathbb{R}^{n-1} + \mathrm{i} V^{\omega}$ make up the covering of T_+.

To define the cohomology we will use the nonstandard language of Radon cocycles.

Definition. We will call the family of holomorphic functions

$$\Phi(\omega, z') = \varphi_{\omega}(\langle \omega, r\rangle, u) , \quad z' = (r, u) \in T^{\omega} ,$$

a Radon–Hardy $(p-2)$-cocycle if

$$\|\Phi\|_{\omega}^2 = \sup_{\rho^2 > \mu^2, \rho > 0} \int_{\mathbb{R}^q} \frac{\partial^{p-2}}{\partial \lambda^{p-2}} \varphi_{\omega}(\sigma + \mathrm{i}\rho, \lambda + \mathrm{i}\mu) \overline{\varphi_{\omega}(\sigma + \mathrm{i}\rho, \lambda + \mathrm{i}\mu)} \, \mathrm{d}\sigma \, \mathrm{d}\lambda < \infty ,$$

$$\|\Phi\|^2 = \int_{\mathrm{S}^{p-2}} \|\Phi\|_{\omega}^2 \, \mathrm{d}\omega < \infty ;$$

$\mathrm{d}\omega$ is the invariant measure on S^{p-1},

$$\mathrm{d}\omega = (|r|^{-p+2} \mathrm{d}|r|) \rfloor \mathrm{d}r , \quad r \in \mathbb{R}^{p-1} .$$

We will call the Hardy space $H_2^{(p-2)}(T_+)$ of the $(p-2)$-dimensional $\bar{\partial}$-cohomology in T_+ the Hilbert space of cocycles Φ.

Remark. For the covering $\{T^{\omega}\}$ the subspace of the L^2 coboundary is trivial.

Proposition. *Let* $f(x') \in \mathscr{L}_+$, $\tilde{f}(\xi') \in \tilde{\mathscr{L}}_+$, *and*

$$f_{\omega}(t, u) = \mathscr{F}^{-1}_{(\rho,\eta)\mapsto(t,u)} \tilde{f}(\rho\omega, \eta)$$

(inverse Fourier transform on variables (ρ, η)*). Then*

$$f \Rightarrow \Phi(\omega; z') = f_{\omega}(\langle \omega, r \rangle, u)$$

is the isometry between the Hilbert spaces $\mathscr{L}_+$ *and* $H_2^{(p-2)}(\mathscr{D}_+)$.

Let

$$\hat{f}^{\omega}(t, u) = \int_{\langle \omega, r \rangle = t} f(r, u)(\langle \omega, \mathrm{d}r \rangle \rfloor \mathrm{d}r \wedge \mathrm{d}u)$$

(the Radon transform in r) and

$$\hat{f}^{\omega} = \hat{f}^{\omega}_{+} + \hat{f}^{\omega}_{-} ,$$

where $\hat{f}^{\omega}_{\pm}$ *is holomorphic on* $t \in \mathbb{C}^1$ *in the upper (lower) halfplane. Then*

$$f_{\omega} = \hat{f}^{\omega}_{+} .$$

The proposition is a direct consequence of the Parseval formula (in spherical coordinates) and the relation between the Radon and Fourier transforms.

5. Čech and Dolbeault cohomologies

We now construct Čech and Dolbeault L^2 cocycles corresponding to the Radon–Hardy cocycles.

Let $\Omega = (\omega^1, ..., \omega^{p-1})$, $\omega^j \in S^{p-2}$, and S_{Ω} be the simplex with vertices in $\omega^1, ..., \omega^{p-1}$.

Proposition 4. *For the Radon–Hardy cocycle* Φ^{ω} *let*

$$\Phi^{\Omega}(z') = \int_{S_{\Omega}} \langle \omega, \partial/\partial r \rangle^{p-2} \Phi^{\omega}(z') \, \mathrm{d}\omega ,$$

$$z' \in T^{\Omega} = T^{\omega^1} \cap \cdots \cap T^{\omega^{p-1}} , \quad z' = (r, u) . \tag{11}$$

Then $\Phi^{\Omega} \in H^2(T^{\Omega})$ *(Hardy space) and* $\{\Phi^{\Omega}\}$ *is a Čech* $(p-2)$*-cocycle,*

$$\sum_{j=1}^{p} (-1)^j \Phi^{\omega_1, ..., \hat{\omega}_j, ..., \omega_p}(z') = 0 , \quad z' \in T^{\omega^1} \cap \cdots \cap T^{\omega^p} . \tag{12}$$

Conversely, if $\{\Phi^{\Omega}\}$ is a *cocycle with elements from* $H^2(T^{\Omega})$ *then* Φ^{Ω} *admits a representation* (11).

The cocycle property (12) for (11) can be verified directly. If $\Phi^{\Omega}\in H^2(T^{\Omega})$ then there is a representation (11) but Φ^{ω} can be dependent on Ω. But the additive property of the integral on S^{p-2} gives the independence from Ω.

Let $\tilde{T}_+$ be the manifold of pairs (ω, z'), $\omega\in S^{p-2}$, $z'\in T^{\omega}$, and let $\pi: \tilde{T}_+\to T_+$ be the natural projection.

Proposition 5. *Let $\omega=\gamma(z')$, $z'\in T^+$, be an arbitrary section of $\pi:\tilde{T}_+\to T_+$ and*

$$\check{\Phi}_\gamma=\{\langle\omega, \partial/\partial r\rangle^{p-1}\Phi(\omega, z')\ \mathrm{d}\omega|_{\omega=\gamma(z')}\}^{(0,p-2)} .$$

(We consider the restriction of the form on $\omega=\gamma(z')$ as the form of T_+ and take its $(0, p-2)$ component.) Then

(i) $\check{\Phi}_\gamma$ is $\bar{\partial}$-closed;

(ii) the cohomology class $\check{\Phi}=\{\check{\Phi}_\gamma\}$ is independent of γ;

(iii) $\check{\Phi}$ corresponds to the Čech class $\{\Phi^{\Omega}\}$;

(iv)

$$\varphi_\omega(t, \omega)=\int_{\substack{\langle\omega,r\rangle=t\\ \mathrm{Im}\ r=\mathrm{const.}}}\check{\phi}_\gamma .$$

The proof uses the inversion formula for the Radon–Penrose transform of $\bar{\partial}$-cohomology (cf. refs. [3,4]).

6. Affine invariant definition of cohomology

Theorem 2 implies that $H_2^{(p-2)}(\mathscr{D}_+)$ for odd n is invariant under the induced action of SO(p, q). But our definition is not invariant under either the conformal group SO(p, q) or the affine group $G_{\mathscr{P}}$. To prove the invariance it is sufficient to extend the covering up to an invariant one. We begin with an affine construction.

Let Γ_+ be the set of q-frames λ,

$$\lambda=\{\lambda^1, ..., \lambda^q\} , \quad \lambda^j\in\mathbb{R}^{n-1} ,$$

such that the intersections V_λ of V_+ by the plane generated by λ has the signature $(1, q-1)$. Then the dual cone V^λ can be defined by the condition

$$s(\lambda; \langle\lambda^1, x'\rangle, ..., \langle\lambda^q, x'\rangle)>0 , \quad \langle\lambda^1, x'\rangle>0 ,$$

where $s(\lambda; \cdot)$ is a quadratic form in $\mathbb{R}^q$, sign $s(\lambda; \cdot)=(1, q-1)$. We have

$$V^\lambda\subset V_+ , \quad V_+=\bigcup_{\lambda\in\Gamma_+} V^\lambda ,$$

and the corresponding covering of T_+ by tube domains T^λ,

$$\lambda \in \Gamma_+ : T_+ = \bigcup_{x \in \Gamma_+} T^{\lambda} .$$

The covering from n.4 corresponds to $S^{p-2} \subset \Gamma_+$,

$$\lambda^1 = (\omega, 0, \ldots, 0) , \qquad \lambda^j = \delta_i^{p-1+j} , \quad 2 \le j \le q .$$

Let

$$\Phi(\lambda; z') = \varphi_\lambda(\langle \lambda^1, z' \rangle, \ldots, \langle \lambda^q, z' \rangle) , \quad \lambda \in \Gamma_+ ,$$

be a family of holomorphic functions in T^λ, satisfying F. John's system of differential equations

$$\partial^2 \varphi_\lambda(t) / \partial\lambda_k^i \, \partial\lambda_l^j = \partial^2 \varphi_\lambda(t) / \partial\lambda_l^i \, \partial\lambda_k^j .$$

Then [5] the differential form

$$\kappa\Phi = \det\left(\lambda^1, \ldots, \lambda^q, \left\{ \sum_{j=1}^{q} \frac{\partial \Phi(\lambda; z')}{\partial \langle \lambda^j, z' \rangle} \, \mathrm{d}\lambda^j \right\}^{p-2} \right)$$

(where we have $(p-2)$ equal columns $\{ \ \}$ and we use the exterior product of differential forms for the calculation of the determinant) is closed.

The corresponding Čech cocycle is given by

$$\Phi^A(z') = \int_{S(A)} \kappa\Phi , \quad A = \lambda^1, \ldots, \lambda^{p-1} , \quad \lambda^j \in \Gamma_+ ,$$

where $S(A)$ is the simplex with vertices $\lambda^1, \ldots, \lambda^{p-1}$. If $\tilde{T}_+ = \{(\lambda, z'), z' \in T^\lambda\}$, $\lambda = \lambda(z')$ is a section of $\tilde{T}_+ \to T_+$, then

$$\check{\Phi} = (\kappa\Phi|_{\lambda = \lambda(z')})^{(0, p-2)}$$

is the corresponding Dolbeault form.

Let

$$(\kappa\Phi, \Phi)(\lambda, \mathrm{d}\lambda) = \sup_{\operatorname{Im} z' \in V^\lambda} \int_{\mathbb{R}^{n-1}} \kappa\Phi \cdot \Phi \, \mathrm{d}(\operatorname{Re} z') .$$

The family Φ is called a Radon–Hardy cocycle if

$$\|\Phi\|^2 = \frac{1}{c(\gamma)} \int_\gamma (\kappa\Phi, \Phi)(\lambda, \mathrm{d}\lambda) < \infty ,$$

where γ is a $(p-2)$-cycle in Γ_+. Let $f(x') \in \mathscr{L}_+$ and

$$\hat{f}(\lambda^1, \ldots, \lambda^q; t^1, \ldots, t^q) = \int_{\mathbb{R}^{n-1}} f(x') \delta(\langle \lambda^1, x' \rangle - t^1) \cdots \delta(\langle \lambda^q, x' \rangle - t^q) \, \mathrm{d}x'$$

be its Radon–John transform. Then

$$\hat{f}(\lambda; t) = \hat{f}_{+}(\lambda; t) + \hat{f}_{-}(\lambda; t) ,$$

where $\hat{f}_{\pm}$ are holomorphic in the tube domains $\{s(\lambda; \operatorname{Im} t) > 0, \operatorname{Im} t_1 \gtrless 0\}$ and $\Phi_\lambda(z') = \hat{f}_{+}(\lambda; \langle \lambda, z' \rangle)$ is a Radon–Hardy cocycle.

Remark about the conformal invariant construction. In our construction the point $x^0 = (0, 1, 0, ..., 0)$ was special. If we fix an arbitrary x^0, $Q(x^0) = 0$, and take the tangent plane to $PH = \{Q(x) = 0\}$ in x^0,

$$\langle \eta, x \rangle = 0 , \quad \eta = (\partial Q / \partial x)(x_0) ,$$

as the plane at infinity, we can repeat our considerations. The union of all coverings $\{T^\lambda\}$ for all x^0, $Q(x^0) = 0$ is an SO(p, q)-invariant covering and in such a way we will get an SO(p, q)-invariant description of $H_2^{(p-2)}(\mathcal{D}_+)$. The description of this construction as well as the decomposition $U_\pm$ on the irreducible representations of SO$(p, q-1)$ will be the subject of a separate paper.

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